Roots of Unity

Moving from arithmetic to algebra, I briefly and mistakenly thought this result could be confirmed by the **second-order** polynomial equation

02.00] $x^2 - 1 = 0$

to which the solutions (**roots**) are of course

02.10] $x = \sqrt[2]{1} = \pm 1$

02.20] More formally, the polynomial can be factored

$$(x - 1)(x + 1) = 0$$

from which the roots can be confirmed by inspection of the brackets.

What about the **third-order** polynomial equation

03.00] $x^3 - 1 = 0$

The roots are of course

03.10] $x = \sqrt[3]{1} = ?$

One root is +1, as always for these odd-orders, but is that all there are ?

03.20] More formally, the polynomial can be factored

$$(x - 1)(x^2 + x + 1) = 0$$

from which the root $x = +1$ can be confirmed by inspection of the first bracket.

03.30] The "missing" roots can, however, be extracted from the second bracket by use of the quadratic solution.

$$x = \frac{-1 \pm \sqrt{(1.1 - 4.1.1)}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

This invokes the imaginary quantity "i", even more surprisingly than -1 had been.

What about the **fourth-order** polynomial equation

04.00] $x^4 - 1 = 0$

The roots are of course

04.10] $x = \sqrt[4]{1} = ?$

Two roots are ±1, as always for even orders, but are these all there are ?

04.20] More formally, the polynomial can be factored

$$(x - 1)(x^3 + x^2 + x + 1) = 0$$

from which the root x = +1 can be seen by inspection of the first bracket, and the root x = -1 can be seen by inspection of the second bracket.

04.30] Reaching into my magicians' top hat (ie googling the answer), I could reveal that there are two remaining roots, $x = \pm i$. But that's not a really satisfactory general procedure.

In fact, as a special case of the **Fundamental Law of Algebra**, the nth roots of 1 (ie **unity**) are n-fold, or in plainer terms, unity has precisely n nth roots – one of which is +1 for odd-orders and two of which are ±1 for even orders.

(Incidentally, the first root of unity is just unity itself, which makes sense.)

05.00] Where are these other roots hiding ? Not quite as in the old pantomime routine "Look behind you !", but "Look to the right of you !" or "Look to the left of you !" – where the **complex numbers** are located.

There are at least three ways of viewing these exotic new entities

05.10] As real numbers r embellished with circular phase-factors $e^{i\varphi}$

$z = re^{i\varphi} = r.\exp(i\varphi)$ … this is the one we want in this context

05.11] Or as quasi-vectorial combinations

$z = x + iy$ where $x = r(\cos\varphi + i \sin\varphi)$

05.12] Or as mysterious marriages of paired real numbers

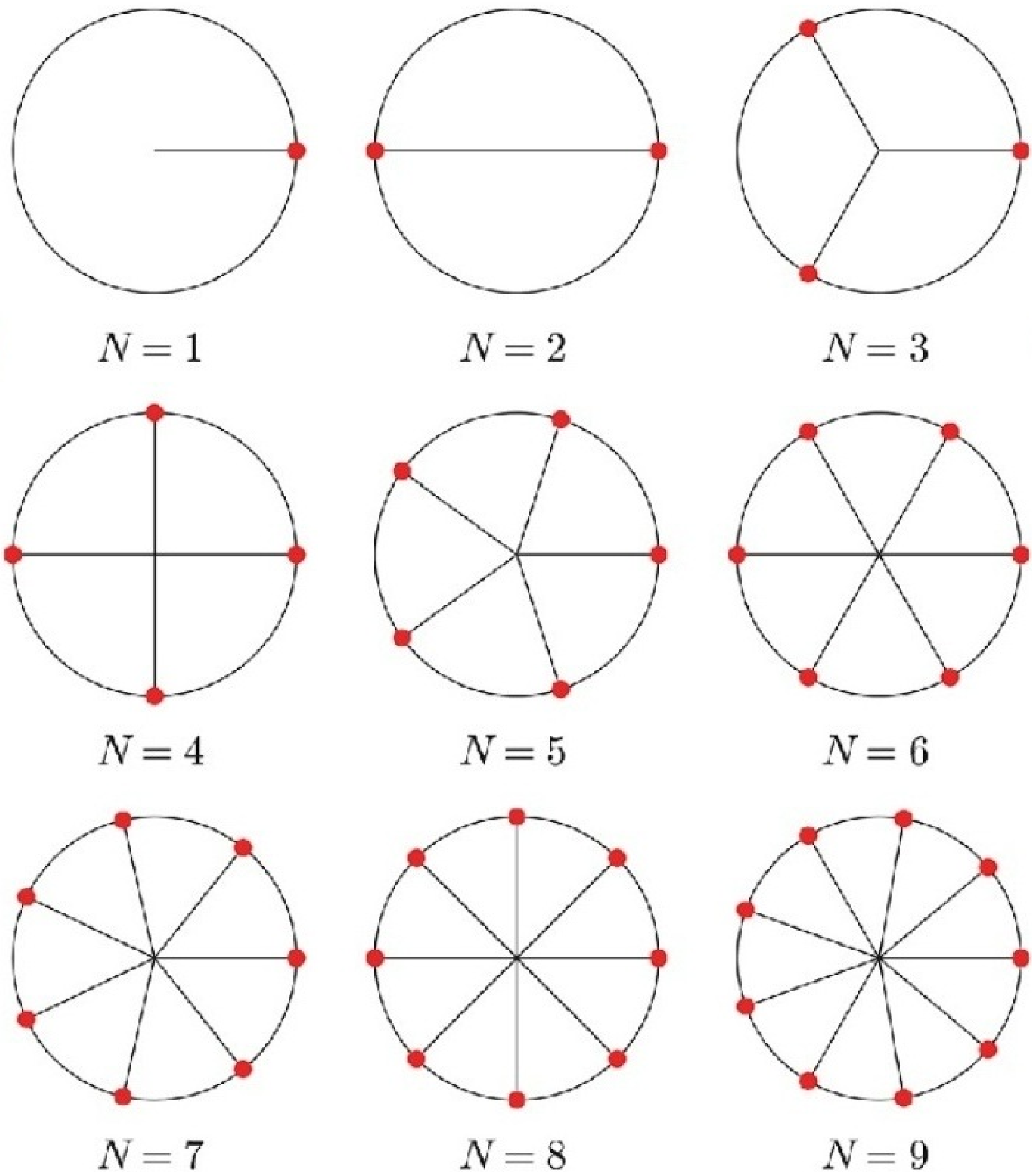
$z = (x, y)$ with very peculiar rules of multiplication

ie $z_1 z_2 = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$

06.00] Very briefly, the n nth roots of unity are equally spaced around a notional circle on a virtual plane (the Argand diagram, first mapped by a mathematical Norwegian surveyor), which is divided into quadrants by the familiar horizontal x-axis and vertical y-axis. Thus they are located at a radius of unity, at regular angular displacements of 2π/n. In fact, the location of each nth root is a vertex of a regular n-polygon, the circumcircle of which touches each vertex.

By convention, the angular starting point in every case is at the intersection of the unit circle with the positive x–axis as in n=!. Click here to see closer detail for cases n=2-7. And please keep in mind that just the one target number w (real or complex) will have an unlimited number of roots depending on how far we care to take N ≡ n.

here = https://www.nagwa.com/en/explainers/257142752623/



Example plots of the distinct Nth roots of unity

Not so easy to visualise, let alone understand, for those of us who never had time or opportunity for complex numbers. What goes around comes around, as Kurt Vonnegut used to remark, and the same is true for complex numbers. As mentioned, they comprise a left-right component x (the familiar bit) and an up-down component y (the unfamiliar bit) and the net effect, for any overall value, is circular.

The tricky bit is that the up-down component is multiplied by the square-root of minus 1. Who ordered that ?

Just as the first-order equation x + 1 = 0 might well have forced mediaeval algebraists to confront the concept of x = -1 (though that wasn't the way it happened), the second-order equation $x^2 + 1 = 0$ might well have forced them to confront that of $x = \sqrt{-1}$, the square-root of minus one (though that wasn't the way it happened either).

It's a neat pedagogical parallel, but they were both historical whoresons. Nevertheless,

05.00] $x + 1 = 0 \Rightarrow x = -1$

05.10] $x^2 + 1 = 0 \Rightarrow x = \sqrt{-1} = i$

Real numbers a and b became associated with the **imaginary number** i to become complex numbers c

05.20] $c = a + ib$

just as **rational numbers** a and b had become attached to **irrational numbers** $\sqrt{n}$ to become **surds** s

05.30] $s = a + b\sqrt{n}$

But what are **real numbers** ? To be honest, they are a hillbilly shotgun shack-up of the **improper rational fractions** and the **mixed numbers** with **fractional parts whose digits neither terminate nor recur.**

Billions of people worldwide use these numbers every day, but they know not what they do. Because, of course, a computer screen (or indeed the computer itself, or a pocket calculator), or a newspaper, inevitably truncates the number of digits, thereby approximating the real numbers to terminating rational numbers.

This is all contemptibly basic stuff to a proper mathematician, who prefers to regard complex numbers as ordered pairs of real numbers (a, b), plus a whole lot of algebraic rules, and real numbers themselves as a rather untidy system of tokens, real or virtual, the unceasing exchange of which has been essential to economic activity since the demise of the barter system.

Few people pause to ask deeper questions such as how can there be a square-root of minus one ? Or a real number that goes on for ever ?

"Well there just is," as my mother would have said. "Little boys shouldn't ask questions !"

In this case, she was possibly on firmer ground than usual, aligning herself with the Platonists who regard truth of all kinds, mathematics included, as residing in an eternal realm existent since before time itself began – insulated from the vagaries of human belief. The peripatetic Erdős had a more concrete image of 'The Book', his almost tangible vision of the virtual compendium of all the mathematical truths there ever have been or ever will be, as doubtless compiled by Nicolas Bourbaki.

But how does this accommodate the possibility, first raised by Gödel, that there are arithmetical propositions that are undecidable (in a man-made context), even by Creator Mundi ? Could the Goldbach Conjecture, for example, be true in one universe but false in another ? Or the Riemann Hypothesis ? My erstwhile colleague mentioned above queried whether there could possibly be a universe in which the truth of Fermat's Last Theorem would be different from in our own (this was before Andrew Wiles came along). Could any (or all) of these be such possibilities ?

However, there are incontrovertible truths that cannot be circumvented in any system of arithmetic, such as the primeness of 119, despite the composite natures of 117, 118, 121 and 122. Some truth is simply objective, as per the perceptive query from the much-maligned Pontius Pilate, whose sympathies clearly lay with the accused rather than the accusers, but had a serious social disturbance to head off. A problem endemic to administrators rather than mathematicians.

Coming back to earth, I'll delve slightly deeper into the mysterious matters of multiple roots in the next section – please don't forget that I'm on a learning-curve too.

We start with Euler's theorem

$$e^{i\theta} = \cos\theta + i\sin\theta$$

so the nth roots of a complex number $w = r\,e^{i\theta}$

$$w = r(\cos\theta + i\sin\theta) \qquad -\pi < \theta \le \pi$$

are $z_1, z_2, z_3, \ldots z_n$, all of which satisfy the equation

$$z_P^{\,n} = w$$

With de Moivre we write for the 'trivial' case

$$z_1 = w^{1/n} = r^{1/n}(\cos\theta + i\sin\theta)^{1/n} = r^{1/n}(\cos\theta/n + i\sin\theta/n)$$

Adding multiples of 2π to θ, which leaves w unchanged, we find for p = 2 – n that

- all values have the same modulus ($r^{1/n}$), which is the radius of a circle whose centre is the origin of the coordinate system
- the roots will be equally spaced around this circle, that is, the arguments of the roots will differ by steps of 2π/n radians.

Thus for the complex number $w = r(\cos\theta + i\sin\theta)$, de Moivre's theorem gives

$$z_P = r^{1/n}\,[\cos(\theta + 2(p-1)\pi/n) + i\sin(\theta + 2(p-1)\pi/n)] \quad (p = 1 – n)$$

The trivial root ($z_1 = 1$) of unity lies at the intersection of the unit circle and the positive real line in an Argand diagram. The arguments of nth roots of unity increase in an arithmetic sequence increasing by 2π/n radians. In an Argand diagram, this means that we can plot the nth roots of unity by starting with 1 and rotating counterclockwise on the unit circle by steps of 2π/n consecutively. If we connect consecutive nth roots of unity with line segments, we will obtain a regular polygon inscribed in the unit circle.

First root (n=1, p=1)

$e^0 = 1$

Second roots (n=2, p=1-2)

$e^0 = 1$, $\exp 2\pi i/2 = \exp \pi = -1$

Third roots (n=3, p=1-3)

$e^0 = 1$, exp $2\pi i/3$, exp $4\pi i/3$,

Fourth roots (n=4, p=1-4)

e^0, exp $2\pi i/4$, exp $4\pi i/4$, exp $6\pi i/4$

Fifth roots (n=5, p=1-5)

e^0, exp $2\pi i/5$, exp $4\pi i/5$, exp $6\pi i/5$, exp $8\pi i/5$

Sixth roots (n=6, p=1-6)

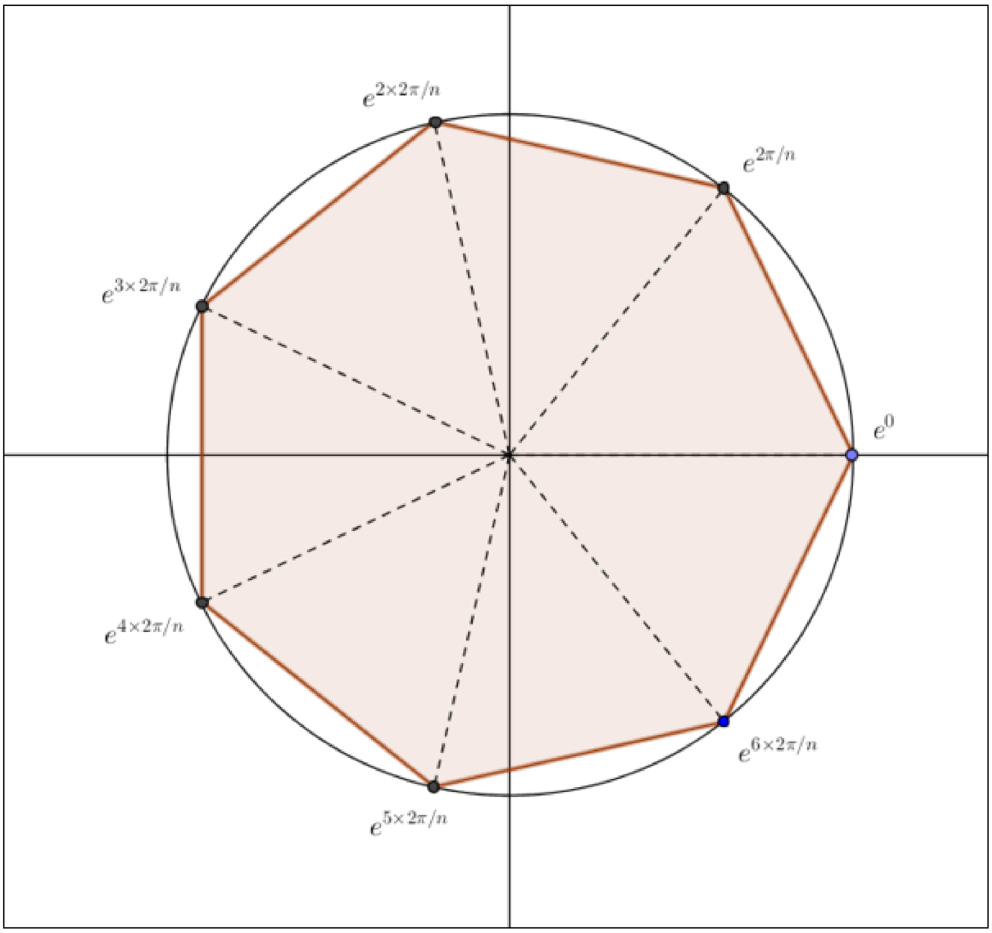
e^0, exp $2\pi i/6$, exp $4\pi i/6$, exp $6\pi i/6$, exp $8\pi i/6$, exp $10\pi i/6$

Seventh roots (n=7, p=1-7)

e^0, exp $2\pi i/7$, exp $4\pi i/7$, exp $6\pi i/7$, exp $8\pi i/7$, exp $10\pi i/7$, exp $12\pi i/7$

(ad infinitum)

https://proofwiki.org/wiki/Complex_Roots_of_Unity_are_Vertices_of_Regular_Polygon_Inscribed_in_Circle



7th roots of unity generate a circumscribed heptagon