The nth roots of minus unity

01.00] From a purely arithmetical starting-point, one might ask, mischievously, what is the **square root** of **minus unity** ?

$$x = \sqrt{(-1)}$$

or equivalently,

$$x^2 = -1$$

The facile answer is of course "i", the mysterious prototype of my father's frequently-aired expression "the square root of minus b*gger-all". But i is far from negligible, and has mathematical status equal to unity itself.

01.10] From an algebraic point of view, we recall Harriott's pioneering insights back in Elizabethan times that the proper way to present this is as a polynomial equation of the form

$$x^2 + 1 = 0$$

And we also know from the Fundamental Theorem of Algebra that this equation has in fact two solutions (as many as the highest degree of the unknown).

01.20] The first step is to factorise the equation, if at all possible. And lo, as Eddington used to say, it is indeed.

$$x^2 - (i)^2 = 0$$

$$(x - i)(x + i) = 0$$

So the **square root** of **minus unity** is not just i but -i also.

02.00] How therefore do we cope with the **cube roots** of **minus unity** ? At least we now know to present the situation as a polynomial equation.

$$x = \sqrt[3]{(-1)}$$

$$x^3 + 1 = 0$$

02.10] x = -1 is clearly one solution, and therefore, by algebraic division (often cited as 'synthetic division', I don't know why) we find

$$x^3 + 1 = (x + 1)(x^2 - x + 1) = 0$$

02.11] The roots of $(x^2 - x + 1)$ can be found from the quadratic formula as

$$\frac{1 \pm \sqrt{[(-1)^2 - 4(1)(1)]}}{2}$$

From which $x = \dfrac{1 \pm i\sqrt{3}}{2}$

02.20] In summary, the **cube roots** of **minus unity** are -1, $\dfrac{1 \pm i\sqrt{3}}{2}$

03.00] Can we progress to the **fourth roots** of **minus unity** ?

$$x = \sqrt[4]{(-1)}$$

$$x^4 + 1 = 0$$

03.10] Not easily. We need to devise a general-purpose algorithm for such higher roots, and de Moivre's theorem is the way forward.

Let $v = s(\cos\alpha + i\sin\alpha)$ be the nth root of $z = r(\cos\theta + i\sin\theta)$

so that $v^n = z$

$\therefore \quad [s(\cos\alpha + i\sin\alpha)]^n = r(\cos\theta + i\sin\theta)$

$\therefore \quad s^n(\cos n\alpha + i\sin n\alpha) = r(\cos\theta + i\sin\theta)$

Equating like with like,

(1) $s = \sqrt[n]{r}$ (immaterial as r = 1 for plus or minus unity)

and

$$\cos n\alpha + i\sin n\alpha = \cos\theta + i\sin\theta$$

Equating real with real and imaginary with imaginary,

$n\alpha = \theta + 2\pi k$ (where k = 0,1, 2, 3, 4, …)

(2) $\alpha = \dfrac{\theta + 2\pi k}{n}$

03.20] So for the fourth roots of minus unity, where $n = 4$ and $\theta = \pi$
(nb that $z = \cos\pi + i\sin\pi = -1$ defines minus unity on the Argand diagram),

$\alpha\ (k=0) = \dfrac{\theta}{4} = \pi/4 = 45^o$ $\quad \cos\alpha = +(\sqrt{2}/2) \quad \sin\alpha = +(\sqrt{2}/2)$

$\alpha\ (k=1) = \dfrac{\theta + 2\pi}{4} = 3\pi/4 = 135^o$ $\quad \cos\alpha = -(\sqrt{2}/2) \quad \sin\alpha = +(\sqrt{2}/2)$

$\alpha\ (k=2) = \dfrac{\theta + 4\pi}{4} = 5\pi/4 = 225^o$ $\quad \cos\alpha = -(\sqrt{2}/2) \quad \sin\alpha = -(\sqrt{2}/2)$

$\alpha\ (k=3) = \dfrac{\theta + 6\pi}{4} = 7\pi/4 = 315^o$ $\quad \cos\alpha = +(\sqrt{2}/2) \quad \sin\alpha = -(\sqrt{2}/2)$

03.30] In summary, the **fourth roots** of **minus unity** are $(\sqrt{2}/2)(1 + i)$,

(√2/2) (– 1 + i), (√2/2) (– 1 – i), (√2/2) (1 – i)

NB the following tabulation, or maybe a more complete one, is immensely useful !

more complete one = https://en.wikipedia.org/wiki/Exact_trigonometric_values

θ		*sin* θ	*cos* θ	*tan* θ	*csc* θ	*sec* θ	*cot* θ
Rad	**Deg**						
0 / 2π	0	0	1	0	Undef	1	Undef
π/6	30	1/2	√3/2	√3/3	2	2√3/3	√3
π/4	45	√2/2	√2/2	1	√2	√2	1
π/3	60	√3/2	1/2	√3	2√3/3	2	√3/3
π/2	90	1	0	Undef	1	Undef	0
2π/3	120	√3/2	− 1/2	− √3	2√3/3	-2	− √3/3
3π/4	135	√2/2	− √2/2	-1	√2	− √2	-1
5π/6	150	1/2	− √3/2	− √3/3	2	− 2√3/3	− √3
π	180	0	-1	0	Undef	-1	Undef
7π/6	210	− 1/2	− √3/2	√3/3	-2	− 2√3/3	√3
5π/4	225	− √2/2	− √2/2	1	− √2	− √2	1
4π/3	240	− √3/2	− 1/2	√3	− 2√3/3	-2	√3/3
3π/2	270	-1	0	Undef	-1	Undef	0
5π/3	300	− √3/2	1/2	− √3	− 2√3/3	2	− √3/3
7π/4	315	− √2/2	√2/2	-1	− √2	√2	-1
11π/6	330	− 1/2	√3/2	− √3/3	-2	2√3/3	− √3