Divisibility is the foundation of number theory (Olson)

The interplay between decimal notation and algebraic symbolism is quite the trickiest aspect of what is otherwise very easy to grasp.

In decimal notation an integer N is represented as

$$N = d_n d_{n-1} \ldots. d_4 d_3 d_2 d_1 d_0$$

which in algebraic symbolism is represented as

$$N = 10^n d_n + 10^{n-1} d_{n-1} + \ldots. + 10^4 d_4 + 10^3 d_3 + 10^2 d_2 + 10^1 d_1 + 10^0 d_0$$

where of course 10^1 is simply 10 and 10^0 is (not quite so simply) unity.

Quite a commonly-used alternative takes the form (for eg a four-figure integer)

$$N = ABCD = 1000A + 100B + 10C + D$$

and (except in a couple of five-figure cases) I'm going to adopt this until we get to the section where modular congruence (oo-er) gets a chance to show off, but basically the job's done by then anyway.

- nb that a|b means that b is divisible by a

- Divisibility by 2

  ABCD = [1000A + 100B + 10C] + D

  2|1000 and 2|100 and 2|10, so […] is divisible by 2.
  So if ABCD is to be divisible by 2, D must also be divisible by 2.
  Hence the final digit D must be divisible by 2 (ie 0,2,4,6,8).

- Divisibility by 3

  ABCD = (999A + A) + (99B + B) + (9C + C) + D

  = [999A + 99B + 9C] + (A + B + C + D)

  3|999 and 3|99 and 3|9, so […] is divisible by 3.
  So if ABCD is to be divisible by 3, (A + B + C + D) must also be divisible by 3.
  Hence the total of all digits ABCD must be divisible by 3.

- Divisibility by 4

  ABCD = [1000A + 100B] + 10C + D

  4|1000 and 4|100, so […] is divisible by 4.
  So if ABCD is to be divisible by 4, 10C + D must also be divisible by 4.
  Hence the two final digits CD must be divisible by 4.

- Divisibility by 5

  ABCD = [1000A + 100B + 10C] + D

  5|1000 and 5|100 and 5|10, so […] is divisible by 5.
  So if ABCD is to be divisible by 5, D must also be divisible by 5.
  Hence the final digit D must be divisible by 5 (ie 0 or 5).

- Divisibility by 7

  See below

- Divisibility by 8

  ABCD = [1000A] + 100B + 10C + D

  8|1000, so […] is divisible by 8.
  So if ABCD is to be divisible by 8, 100B + 10C + D must also be divisible by 8.
  Hence the three final digits BCD must be divisible by 8.

- Divisibility by 9

  ABCD = (999A + A) + (99B + B) + (9C + C) + D

  = [999A + 99B + 9C] + (A + B + C + D)

  9|999 and 9|99 and 9|9, so […] is divisible by 9.
  So if ABCD is to be divisible by 9, (A + B + C + D) must also be divisible by 9.
  Hence the total of all digits ABCD must be divisible by 9.

- Divisibility by 10

  ABCD = [1000A + 100B + 10C] + D

  10|1000 and 10|100 and 10|10, so […] is divisible by 10.
  So if ABCD is to be divisible by 10, D must also be divisible by 10.
  Hence the final digit D must be divisible by 10 (ie 0).

- Divisibility by 11

  ABCD = (1001A – A) + (99B + B) + (11C – C) + D

  = [1001A + 99B + 11C] – (A – B + C – D)

  11|1001 and 11|99 and 11|11, so […] is divisible by 11.
  So if ABCD is to be divisible by 11, (A – B + C – D) must also be divisible by 11.
  Hence the total of alternately signed digits ABCD must be divisible by 11.

  BUT !!!

  ABCDE = (9999A + A) + (1001B – B) + (99C + C) + (11D – D) + E

  = [9999A + 1001B + 99C + 11D] + (A – B + C – D + E)

  11|9999 and 11|1001 and 11|99 and 11|11, so […] is divisible by 11.
  So if ABCDE is to be divisible by 11, (A – B + C – D) must also be divisible by 11.
  Hence the total of alternately signed digits ABCDE must be divisible by 11.

  THE CASES ALTERNATE BUT THE RULE IS CONSTANT

- Divisibility by 7, 13, 17 or 19

  See below

- Divisibility by 16

  ABCDE = [10000A] + 1000B + 100C + 10D + E

16|10000, so […] is divisible by 16.
So if ABCDE is to be divisible by 16, 1000B + 100C + 10D + E must also be divisible by 16.

Hence the four final digits BCDE must be divisible by 16.

Divisibility testing by multiple factors N/ab… is of course permissible, provided that the factors a, b ,,, are all coprime with each other

- Divisibility by 6 as per 2 and 3

- Divisibility by 12 as per 3 and 4

- Divisibility by 14 as per 2 and 7

- Divisibility by 15 as per 3 and 5

- Divisibility by 7

http://mathandmultimedia.com/2012/02/29/divisibility-by-7-and-its-proof/

https://artofproblemsolving.com/wiki/index.php/Divisibility_rules#Divisibility_Rule_for_7

As mentioned earlier, $N = d_n d_{n-1} \dots d_4 d_3 d_2 d_1 d_0$
$= 10\, d_n d_{n-1} \dots d_4 d_3 d_2 d_1 + d_0$

which for simplicity we write as 10a + b

1. The divisibility rule for 7 is that a – 2b should be divisible by 7

2. and that if N is divisible by 7 then 10a + b is divisible by 7

   and is a masterpiece of ingenuity :

First we prove that (surely this is sufficient ?)

1. If a – 2b is divisible by 7 then a – 2b = 7k where k is some integer,

   Multiply by 10 and add b: 10a – 20b + b = 70k + b

   so: 10a + b = 70k +b + 20b
   ie: 10a + b = 70k + 21b
   ie: N = 7(k + 3b)

   So N is self-evidently divisible by 7

Next we note that (though is this really necessary ?)

2. If N = 10a + b is divisible by 7 then 10a + b = 7k where k is some integer,

   Subtract 21b: 10a + b – 21b = 7k – 21b

So: $10a - 20b = 7k - 21b$
Ie: $10(a - 2b) = 7(k - 3b)$

So $a - 2b$ is self-evidently divisible by 7

- Divisibility by 13

Once again N is written as $10a + b$

The divisibility rule for 13 is that $a + 4b$ should be divisible by 13

1. If $a + 4b$ is divisible by 13 then $a + 4b = 13k$ where k is some integer,

   Multiply by 10 and add b: $10a + 40b + b = 130k + b$

   so $10a + b = 130k + b - 40b$
   Ie: $10a + b = 130k - 39b$
   Ie: $N = 13(10k - 3b)$

   So N is self-evidently divisible by 13

- Divisibility by 17

Once again N is written as $10a + b$

The divisibility rule for 17 is that $a - 5b$ should be divisible by 17

1. If $a - 5b$ is divisible by 17 then $a - 5b = 17k$ where k is some integer,

   Multiply by 10 and add b: $10a - 50b + b = 170k + b$

   so: $10a + b = 170k + b + 50b$
   Ie: $10a + b = 170k + 51b$
   Ie: $N = 17(10k + 3b)$

   So N is self-evidently divisible by 17

- Divisibility by 19

Once again N is written as $10a + b$

The divisibility rule for 19 is that $a + 2b$ should be divisible by 19

1. If $a + 2b$ is divisible by 19 then $a + 2b = 19k$ where k is some integer,

   Multiply by 10 and add b: $10a + 20b + b = 190k + b$

   so: $10a + b = 190k + b - 20b$
   Ie: $10a + b = 190k - 19b$
   Ie: $N = 19(10k - b)$

   So N is self-evidently divisible by 19

- Divisibility by **primes in general** (7, 11, 13 etc)

https://artofproblemsolving.com/wiki/index.php/Divisibility_rules#Divisibility_Rule_for_7

For every prime number other than 2 and 5, there exists a rule similar to [this rule] for divisibility by 7. For a general prime p, there exists some number q such that an integer is divisible by p if and only if truncating the last digit, multiplying it by q and subtracting it from the remaining number gives us a result divisible by p.

- The divisibility rule 2 for 7 says that for p=7, q=2.
- The divisibility rule for 11 is equivalent to choosing q = 1.
- The divisibility rule for 3 is equivalent to choosing q = -1.

These rules can also be found under the appropriate conditions in other than 10. Also note that these rules exist in two forms: if q is replaced by p – q then subtraction may be replaced with addition. We see one instance of this in the divisibility rule for 13: we could multiply by 9 and subtract rather than multiplying by 4 and adding.

Divisibility Rule for 13

Rule 1: Truncate the last digit, multiply it by 4 and add it to the rest of the number. The result is divisible by 13 if and only if the original number was divisble by 13. This process can be repeated for large numbers, as with the second divisibility rule for 7.

Proof

Rule 2: Partition N into 3 digit numbers from the right ($d_3d_2d_1, d_6d_5d_4, \cdots$). The alternating sum ($d_3d_2d_1 - d_6d_5d_4 + d_9d_8d_7 - \cdots$) is divisible by 13 if and only if N is divisible by 13.

Proof

Divisibility Rule for 17

Truncate the last digit, multiply it by 5 and subtract from the remaining leading number. The number is divisible if and only if the result is divisible. The process can be repeated for any number.

Proof

Divisibility Rule for 19

Truncate the last digit, multiply it by 2 and add to the remaining leading number. The number is divisible if and only if the result is divisible. This can also be repeated for large numbers.

- **Enough already…**

To be honest, I don't yet understand the next procedure. But am I bovvered ? There are so many divisibility procedures, penny plain or tuppence coloured, that one might as well choose an easy one.

https://www.johndcook.com/blog/2020/11/10/test-for-divisibility-by-13/

Testing divisibility by 7, 11, and 13

We're going to **kill three birds with one stone** by presenting a rule for testing divisibility by 13 that also gives new rules for testing divisibility by 7 and 11. So if you're trying to factor a number by hand, this will give a way to test three primes at once.

- To test divisibility by 7, 11, and 13, write your number with digits grouped into threes as usual. For example, 11,037,989

Then think of each group as a separate number — e.g. 11, 37, and 989 — and take the alternating sum, starting with a + sign on the last term, 989 – 37 + 11

The original number is divisible by 7 (or 11 or 13) if this alternating sum is divisible by 7 (or 11 or 13 respectively).

The alternating sum in our example is 963, which is clearly 9*107, and not divisible by 7, 11, or 13. Therefore 11,037,989 is not divisible by 7, 11, or 13.

Here's another example.

- Let's start with 4,894,498,518

  The alternating sum is 518 – 498 + 894 – 4 = 910

  The sum takes a bit of work, but less work than dividing a 10-digit number by 7, 11, and 13.

  The sum 910 factors into 7*13*10, and so it is divisible by 7 and by 13, but not by 11. That tells us 4,894,498,518 is divisible by 7 and 13 but not by 11.

Why this works

The heart of the method is that 7*11*13 = 1001. If I subtract a multiple of 1001 from a number, I don't change its divisibility by 7, 11, or 13. More than that, I don't change its **remainder** by 7, 11, or 13.

The steps in the method amount to adding or subtracting multiples of 1001 and dividing by 1000. The former doesn't change the remainder by 7, 11, or 13, but the latter multiplies the remainder by -1, hence the alternating sum. (1000 is congruent to -1 mod 7, mod 11, and mod 13.) See more formal argument in footnote [1].

So not only can we test for divisibility by 7, 11, and 13 with this method, we can also find the remainders by 7, 11, and 13. The original number and the alternating sum are congruent mod 1001, so they are congruent mod 7, mod 11, and mod 13.

In our first example, n = 11,037,989 and the alternating sum was m = 963. The remainder when m is divided by 7 is 4, so the remainder when n is divided by 7 is also 4. That is, m is congruent to 4 mod 7, and so n is congruent to 4 mod 7. Similarly, m is congruent to 6 mod 11, and so n is congruent to 6 mod 11. And finally m is congruent to 1 mod 13, so n is congruent to 1 mod 13.