HCF and LCM (Burton, pp31-32)

One of my father's favourite obiter dicta was that one damn fool can ask questions that twenty wise men can't answer (such as why is the sky dark at night, for example, or what do women really want ?).

And in my own experience there are many more that I myself would like to ask, and given time will do so.

That aside, there were innumerable concepts in maths that were introduced in my schooldays which simply didn't pass my "So What ?" test. Why did we need to know about them ? Gradually I've realised that there were reasons that even the maths masters didn't necessarily appreciate.

Maxima and minima, for example, who cared ? It wasn't until I encountered a question in our textbook about a motorcyclist (probably called Paul) who needed to find the fastest route to his destination (probably Damascus) via a mix of cross-country and motorway, that the revelation came upon me – this was a deeply serious technique, a Principle of Least Time (as I subsequently realised) being a recurrent thread in the tapestry of the universe.

So too were (at an earlier stage) Highest Common Factors (Greatest Common Divisors) and Least Common Multiples (Lowest Common Denominators). Was I bovvered ?

- The HCF of two numbers a and b is the largest number that will divide them both.

- The LCM of two numbers a and b is the least (ie smallest) number that that they both divide.

Actually, yes I was bovvered, because these concepts did make the manipulation of fractions immensely easier, as you're probably well aware anyway. But something I noticed along the way was that the product of the HCF and the LCM of two numbers always seemed to equal the product of the numbers themselves.

$$\text{HCF}(a,b) \times \text{LCM}(a,b) = ab$$

- So if a=8 and b=12, then HCF=4 and LCM=24, so HCF x LCM = 96 = ab
- And if a=7 and b=12, then HCF=1 and LCM=84, so HCF x LCM = 84 = ab

Was it always true ? And if so, why ? The time has come to sort this out.

01.00] Suppose that a and b share in toto the prime factors $\{p_1 , \dots , p_r\}$

$a = p_1\hat{\ }k_1 \times p_2\hat{\ }k_2 \times \dots \times p_r\hat{\ }k_r$

$b = p_1\hat{\ }\ell_1 \times p_2\hat{\ }\ell_2 \times \dots \times p_r\hat{\ }\ell_r$

01.10] So $\text{HCF}(a,b) = p_1\hat{\ }\min(k_1,\ell_1) \times p_2\hat{\ }\min(k_2,\ell_2) \times \dots \times p_r\hat{\ }\min(k_r,\ell_r)$

And $\text{LCM}(a,b) = p_1\hat{\ }\max(k_1,\ell_1) \times p_2\hat{\ }\max(k_2,\ell_2) \times \dots \times p_r\hat{\ }\max(k_r,\ell_r)$

Now come two statements that need quite careful thought

- For the HCF you need to pick the <u>lower</u> power of each prime factor
- For the LCM you need to pick the <u>higher</u> power of each prime factor

01.20] So HCF(a,b) x LCM(a,b) =

p_1^[min(k_1,ℓ_1) + max(k_1,ℓ_1)] x … x p_r^[min(k_r,ℓ_r) + max(k_r,ℓ_r)]

02.00] But min(k+ℓ) + max(k+ℓ) = (k+ℓ)

So HCF(a,b) x LCM(a,b) = p_1^[k_1+ ℓ_1] x … x p_r^[k_r + ℓ_r)]

= [p_1^k_1 x … x p_r^k_r] x [p_1^ℓ_1 x … x p_r^ℓ_r]

= ab QED

03.00] So the LCM of a and b is their product divided by their HCF

LCM(a,b) = ab / HCF(a,b)

03.10] All very well, you may say, but first we need to know HCF(a,b). How can that be found ? The obvious answer is to list all the divisors of a and b, and look for the greatest one they have in common. However, this requires a and b to be factorised, not necessarily an easy task.

03.20] Fortunately the ancient Greeks worked out a procedure by which the HCF of any two numbers could be identified without needing to factorise them ! This was set forth by Euclid (though he may well not have originated it) and is known as Euclid's algorithm, or the Euclidean algorithm. There are in fact two basic variants of the algorithm, one based on successive **subtractions**, the other based on successive **divisions**.

For simplicity, let's suppose that a>b, as HCF(a,a) = a.

04.00] The **subtraction** algorithm is based on the identity HCF(a,b) = HCF(b,a – b)

- Let d = HCF(a,b), where (by definition) d divides a and d divides b

- As a result, d also divides a – b
  so that d is also a CF of b and a – b
  but not necessarily the **highest** CF of b and a – b

- Let c be an arbitrary CF of b and a – b
  so that c divides both b and a
  and is therefore also a CF of a and b,
  but not the highest CF of a and b, as d is the highest, so c ≤ d

- So d is certainly the highest CF of b and a – b
  and HCF(a,b) = HCF(b,a – b) QED.

04.10] Let's try a worked example. Suppose we wish to find HCF(27,33).

HCF(27,33) ≡ HCF(33,27) [so that a>b]

HCF(33,27) = HCF(27,33 – 27) = HCF(27,6)

HCF(27,6) = HCF(6,27 – 6) = HCF(6,21) ≡ HCF(21,6)

HCF(21,6) = HCF(6,21 – 6) = HCF (6,15) ≡ HCF(15,6)

HCF(15,6) = HCF(6,15 – 6) = HCF(6,9) ≡ HCF(9,6)

HCF(9,6) = HCF(6,9 – 6) = HCF(6,3)

HCF(6,3) = HCF(3,6 – 3) = HCF(3,3)

HCF(3,3) = HCF(3,3-3) = HCF(3,0) = 3

ie HCF(27,33) = 3, a result we could have predicted at a glance.

04.20] The workings could be fast-tracked through repeated subtractions of the same number, grouped as a **multiple**,

(27,33) = (33,27) -> (27,6) -> (9,6) -> (6,3) -> (3,0)

especially when a and b have widely disparate values

05.00] The **division** algorithm is based on the identity HCF(a,b) = HCF(b, r) where a = qb + r

- Let d = HCF(a,b), where (by definition) d divides a and d divides b
- As a result, d also divides a – qb
  and (by implication) d must also divide r.
  Thus d is a CF of both b and r
  but not necessarily the **highest** CF of b and r
- Let c be an arbitrary CF of b and r
  then c divides (qb + r), and therefore divides a also
  and is therefore also a CF of a and b,
  but not the highest CF of a and b, as d is the highest, so c ≤ d
- So d is certainly the highest CF of b and r
  and HCF(a,b) = HCF(b,r) QED.

Note that r is evaluated as mod(a,b) ie the remainder when a is divided by b.

05.10 Let’s try a worked example. Suppose we wish to find HCF(27,33).

- First, we divide the bigger one by the smaller one to get the remainder:

  33 = 1×27 + 6
  So HCF(33,27) = HCF(27,6)
- Repeating the process:

  27 = 4×6 + 3
  So HCF(27,6) = HCF(6,3)
- And again:

  6 = 2×3 + 0
  So HCF(6,3) = HCF(3,0) = 3

So the same result before, but very much quicker.

05.20 Let's try another example. Suppose we wish to find HCF(12378,3054). The procedure can be stripped to the barest essentials.

(12378, 3054) = 4 x 3054 + 162

(3054, 162) = 18 x 162 + 138

(162, 138) = 1 x 138 + 24

(138, 24) = 5 x 24 + 18

(24, 18) = 1 x 18 + 6

(18, 6) = 3 x 6 + 0

So HCF(12378,3054) = HCF(6,0) = 6

05.30] The division algorithm could be formalised algebraically, but the WYSIWYG principle tells us that the more vivid the example the less need there is for explanation, and I think this is a prime example.

06.00] The LCM appears nowadays to be the poor relation of the HCF, but in fact it was by far the more useful in everyday algebra at secondary school, when at every turn one needed to add or subtract fractions. The first thing you had to do was to express them both (or all) in terms of a **common denominator**, and scale the **numerators** accordingly.

Thus to add halves, thirds and ninths, for example, you had first to express them in terms of eighteenths, as the LCM of 2, 3 and 9 was (and still is) 18. In those days it was called the **Lowest Common Denominator** for that reason. You were supposed to examine each denominator for its prime factors (2, 3, and 3x3, and then work out the LCM as the product of the highest multiples of each prime factor (2 and 3^2 in this case).

Well, of course, we (almost) all let our back-brain come up with the answer. Those without a back-brain, such as me, just multiplied 2, 3 and 9 and came up with 54. This had disadvantages when dealing with denominators such as 32 and 48, for example, where instead of a/32 ± b/48 = (3a ± 2b)/96 we let ourselves in for a denominator of 32 x 48 = 1536, a 16-fold over-kill.

Algebraically, of course, that was according to the text-book, which said that

a/b ± c/d = (ad ± bc)/bd

rather than

a/b ± c/d = (ad ± bc) . [LCM(b,d)/bd] / LCM(b,d)

Eventually I acquired a back-brain, and with it a degree of common sense – in maths, just as in life itself, there are some things you just have to learn through painful experience.