i’th root of i (Paul J Nahin #2 p1)

The nineteenth-century Harvard mathematician Benjamin Peirce (1808-1880) made a tremendous impression on his students. As one of them wrote many years after Peirce’s death, “The appearance of Professor Benjamin Peirce, whose long grey hair, straggling grizzled beard and unusually bright eyes sparkling under a soft felt hat, as he walked briskly but rather ungracefully across the college yard, fitted very well with the opinion current amongst us that we were looking upon a real live genius, who had a touch of the prophet in his make-up.”

That same former student went on to recall that during one lecture “he established the relation connecting π, e and i, $e^{\pi/2} = \sqrt[i]{i}$, which evidently had a strong hold upon his imagination. He dropped his chalk and eraser, put his hands in his pockets, and after contemplating the formula a few minutes turned to his class and said very slowly and impressively, “Gentlemen, that is surely true, it is absolutely paradoxical, we can’t understand it, and we don’t know what it means, but we have proved it, and therefore we know it must be the truth.”

Well we know what to do with it, but that doesn’t mean that we understand what it means intrinsically (rather like ones first girlfriend). We do know what a square (ie second) root is, and a cube (ie third) root is, and so on, but i is not a counting number, so what does i’th mean ?

We can transform it into something more familiar, however.

- Take the i’th power of both sides $[e^{\pi/2}]^{i} = [\sqrt[i]{i}]^{i}$
  So that $e^{i\pi/2} = i$

- Then square both sides $[e^{i\pi/2}]^{2} = i^{2}$
  So that $e^{i\pi} = -1$ aka $e^{i\pi} + 1 = 0$

And so we retrieve Euler’s famous identity.

Which in turn is simply a special case of $e^{i\theta} = \cos\theta + i \sin\theta$, with θ = -180 degrees.
And so at last that’s something we think we do understand !