Shanks transformation

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In numerical analysis, the **Shanks transformation** is a non-linear series acceleration method to increase the rate of convergence of a sequence. This method is named after Daniel Shanks, who rediscovered this sequence transformation in 1955. It was first derived and published by R. Schmidt in 1941.[1]

> One can calculate only a few terms of a perturbation expansion, usually no more than two or three, and almost never more than seven. The resulting series is often slowly convergent, or even divergent. Yet those few terms contain a remarkable amount of information, which the investigator should do his best to extract.
> This viewpoint has been persuasively set forth in a delightful paper by Shanks (1955), who displays a number of amazing examples, including several from fluid mechanics.
>
> Milton D. Van Dyke (1975) *Perturbation methods in fluid mechanics*, p. 202.

Contents

Shanks transformation [edit]

Formulation [edit]

For a sequence $\{a_m\}_{m\in\mathbb{N}}$ the series

$$A = \sum_{m=0}^{\infty} a_m$$

is to be determined. First, the partial sum A_n is defined as:

$$A_n = \sum_{m=0}^{n} a_m$$

and forms a new sequence $\{A_n\}_{n\in\mathbb{N}}$. Provided the series converges, A_n will approach in the limit to A as $n \to \infty$. The Shanks transformation $S(A_n)$ of the sequence A_n is defined as[2][3]

$$S(A_n) = \frac{A_{n+1}\,A_{n-1} - A_n^2}{A_{n+1} - 2A_n + A_{n-1}}$$

and forms a new sequence. The sequence $S(A_n)$ often converges more rapidly than the sequence A_n. Further speed-up may be obtained by repeated use of the Shanks transformation, by computing $S^2(A_n) = S(S(A_n)), S^3(A_n) = S(S(S(A_n)))$, etc.

Note that the non-linear transformation as used in the Shanks transformation is of similar form as used in Aitken's delta-squared process. But while Aitken's method operates on the coefficients $\{a_m\}$ of the original sequence, the Shanks transformation operates on the partial sums A_n.

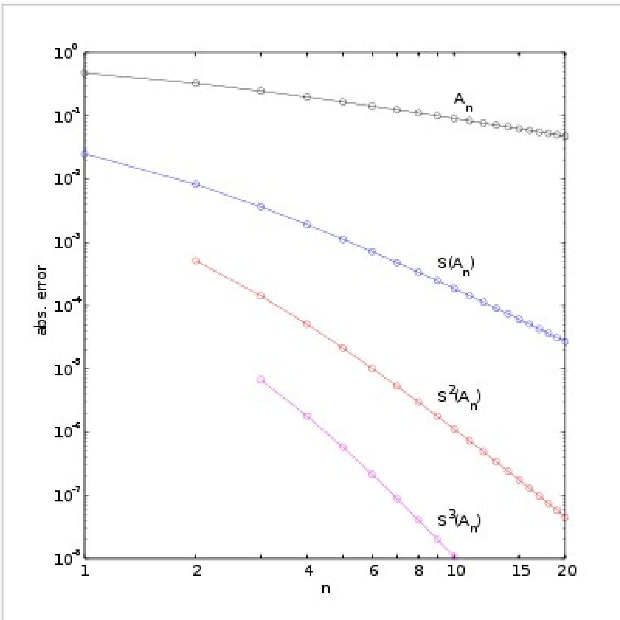
Example [edit]

As an example, consider the slowly-convergent series[3]

$$4\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+1} = 4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots\right)$$

which has the exact sum π ≈ 3.14159265. The partial sum A_6 has only one digit accuracy, while six-figure accuracy requires summing about 400,000 terms.

In the table below, the partial sums A_n, the Shanks transformation $S(A_n)$ on them, as well as the repeated Shanks transformations $S^2(A_n)$ and $S^3(A_n)$ are given for n up to 12. The figure to the right shows the absolute error for the partial sums and Shanks transformation results, clearly showing the improved accuracy and convergence rate.



Absolute error as a function of n in the partial sums A_n and after applying the Shanks transformation once or several times: $S(A_n), S^2(A_n)$ and $S^3(A_n)$. The series used is $4\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots\right)$, which has the exact sum π.

n	A_n	$S(A_n)$	$S^2(A_n)$	$S^3(A_n)$
0	4.00000000	—	—	—
1	2.66666667	3.16666667	—	—
2	3.46666667	3.13333333	3.14210526	—
3	2.89523810	3.14523810	3.14145022	3.14159936
4	3.33968254	3.13968254	3.14164332	3.14159086
5	2.97604618	3.14271284	3.14157129	3.14159323
6	3.28373848	3.14088134	3.14160284	3.14159244
7	3.01707182	3.14207182	3.14158732	3.14159274
8	3.25236593	3.14125482	3.14159566	3.14159261

9	3.04183962	3.14183962	3.14159086	3.14159267
10	3.23231581	3.14140672	3.14159377	3.14159264
11	3.05840277	3.14173610	3.14159192	3.14159266
12	3.21840277	3.14147969	3.14159314	3.14159265

The Shanks transformation $S(A_1)$ already has two-digit accuracy, while the original partial sums only establish the same accuracy at A_{24}. Remarkably, $S^3(A_3)$ has six digits accuracy, obtained from repeated Shank transformations applied to the first seven terms A_0 till A_6. As said before, A_n only obtains 6-digit accuracy after about summing 400,000 terms.

Motivation

[edit]

The Shanks transformation is motivated by the observation that — for larger n — the partial sum A_n quite often behaves approximately as[2]

$$A_n = A + \alpha q^n,$$

with $|q| < 1$ so that the sequence converges transiently to the series result A for $n \to \infty$. So for $n-1, n$ and $n+1$ the respective partial sums are:

$$A_{n-1} = A + \alpha q^{n-1} \quad , \qquad A_n = A + \alpha q^n \qquad \text{and} \qquad A_{n+1} = A + \alpha q^{n+1}.$$

These three equations contain three unknowns: A, α and q. Solving for A gives[2]

$$A = \frac{A_{n+1} A_{n-1} - A_n^2}{A_{n+1} - 2A_n + A_{n-1}}.$$

In the (exceptional) case that the denominator is equal to zero: then $A_n = A$ for all n.

Generalized Shanks transformation

[edit]

The generalized *k*th-order Shanks transformation is given as the ratio of the determinants:[4]

$$S_k(A_n) = \frac{\begin{vmatrix} A_{n-k} & \cdots & A_{n-1} & A_n \\ \Delta A_{n-k} & \cdots & \Delta A_{n-1} & \Delta A_n \\ \Delta A_{n-k+1} & \cdots & \Delta A_n & \Delta A_{n+1} \\ \vdots & & \vdots & \vdots \\ \Delta A_{n-1} & \cdots & \Delta A_{n+k-2} & \Delta A_{n+k-1} \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & 1 & 1 \\ \Delta A_{n-k} & \cdots & \Delta A_{n-1} & \Delta A_n \\ \Delta A_{n-k+1} & \cdots & \Delta A_n & \Delta A_{n+1} \\ \vdots & & \vdots & \vdots \\ \Delta A_{n-1} & \cdots & \Delta A_{n+k-2} & \Delta A_{n+k-1} \end{vmatrix}},$$

with $\Delta A_p = A_{p+1} - A_p$. It is the solution of a model for the convergence behaviour of the partial sums A_n with k distinct transients:

$$A_n = A + \sum_{p=1}^{k} \alpha_p q_p^n.$$

This model for the convergence behaviour contains $2k+1$ unknowns. By evaluating the above equation at the elements $A_{n-k}, A_{n-k+1}, \cdots, A_{n+k}$ and solving for A, the above expression for the kth-order Shanks transformation is obtained. The first-order generalized Shanks transformation is equal to the ordinary Shanks transformation: $S_1(A_n) = S(A_n)$.

The generalized Shanks transformation is closely related to Padé approximants and Padé tables.[4]

See also

[edit]

- Aitken's delta-squared process
- Rate of convergence
- Richardson extrapolation
- sequence transformation

Notes

[edit]

1. ^ Weniger (2003).
2. ^ a b c Bender & Orszag (1999), pp. 368–375.
3. ^ a b Van Dyke (1975), pp. 202–205.
4. ^ a b Bender & Orszag (1999), pp. 389–392.

References

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