Number

There have been so many developments of this concept in Europe since the Middle Ages and I'm sticking mostly with those that meant (or were supposed to mean) something to me at secondary school, or in some cases since then. This is all purely personal – it's not a history, or a tutorial. Some of my statements may be wrong.

01.00] The earliest numbers were simply a tally of objects vital to survival: arrow heads, prey animals successfully slain, and so on. And of course these **counting numbers** were **whole numbers** – a half-dead mammoth didn't qualify. It's often said that these **natural numbers** comprised just {1, 2, 3 …}

01.10] But I think the concept of "none", implying as it did and still does, "not one", bugger-all, meant something then – the thumb and fingers of the left hand firmly closed – even if the mathematical concept of "zero" was yet to emerge.

01.20] It's alleged that "one, two, many" was as far as most primitive people could manage. But I think that qualitative distinctions between "few" and "several" would have been noticeable then, especially to ones wife, even if so many people get them wrong these days – the one implying paucity and the other implying sufficiency. So that takes us from "two or three" to possibly "four or five", the limit of what one relatively unmutilated left hand could display.

02.00] The introduction of zero by the Indian and Arabic civilisations during the western Dark Ages gave rise to the **integers** {0, 1, 2, 3 …}

02.10] The mathematician Kronecker asserted that the Good Lord made the integers and Man made the rest (though he detested Cantor's infinities, of course). It's ironic that Cantor was also the foremost pioneer of set theory, which can provide an ingenious rationale of how the Good Lord set about the task.

02.11] He (ie the GL) started with 'zero' and identified it as the empty set, denoted these days as { } or Ø, according to personal preference. Thitherto, zero had been undefined.

$$0 = \{\ \} = \varnothing$$

02.12] He then identified 'one' as the set containing zero, ie the set containing the empty set. Thitherto, one had been undefined.

$$1 = \{0\} = \{\varnothing\}$$

02.13] He then identified the previously undefined 'two' as the set containing zero and one.

2 = {0, 1} = {Ø, {Ø}}

02.14] He then identified the previously undefined ‘three’ as the set containing zero, one and two.

3 = {0, 1, 2} = {Ø, {Ø}, {Ø, {Ø}}}

02.15] He then identified the previously undefined ‘four’ as the set containing zero, one, two, and three.

4 = {0, 1, 2, 3} = {Ø, {Ø}, {Ø, {Ø}}, {Ø, {Ø}, {Ø, {Ø}}}}

02.16] And so on to infinity; integers are the ultimate free lunch.

03.00] And the acceptance of negative numbers in the Middle Ages gave rise to the concept of **positive integers** and **negative integers** (…, -3, -2, -1, 0, +1, +2, +3, …)

04.00] The re-introduction of the Ancient Greek concept of fractions in the early Middle Ages, followed by the flight of scholars from Constantinople in the mid-15th century, really kick-started the European mathematical upsurge.

Fractions as first envisaged were signed ratios of integers a/b, known as **rational numbers**, or as **vulgar (**aka **common) fractions.** It was implicitly assumed that a and b were **coprime**, ie had no factors in common with one another.

04.10] **Proper fractions** have a<b, and **improper fractions** have a>b. If a=b, of course, the result is the integer 1. If b is an **submultiple** (**aliquot)** of a, the result is an integer of higher value than 1.

04.20] **Improper fractions** can be represented as a **quotient** followed by a proper fraction known as the **remainder.** Such a combination is known as a **mixed number**.

04.21] The evaluation of the **fractional remainder** (by long division) will either **terminate** or **recur**, whatever the particular number-scale.

For a fraction to terminate (rather than recur) in a particular number scale, all the prime factors of its rational denominator (b) must divide the base of that number-scale. In the denary (base 10) scale, for example, the prime factors must be 2 and/or 5).

A fraction which neither terminates nor recurs is therefore called an **irrational number**. If it’s irrational in one number-scale it’s irrational in all other number scales. This isn’t too difficult to prove.

04.22] The number (d) of digits in a **terminating fraction** depends on the value of the base of the particular **number-scale** that is adopted.

04.23] The period of recurrence (ie the number (d) of digits per cycle of recurrence, aka **repetend)** in a **recurrent** fraction likewise depends on the value of the base of the particular number-scale adopted.

04.30] Combinations $a \pm \sqrt{b}$ of integers with one or more irrationals (usually square roots, or less commonly higher-order roots, known collectively as **radicals**) were first encountered in the solution of quadratic equations, and are known as **surds**.

05.00] The **real numbers** can be sub-categorised as follows.

05.10] A **rational number** a/b can be regarded as the solution of a first-degree (ie linear, monomial) equation $x - a/b = 0$, and is therefore **algebraic**.

05.11] An **irrational number** may well still qualify as an **algebraic** number.

05.20] In addition to numbers that are solutions to first-degree equations (rational numbers) or second-, third- and fourth-degree equations (surds and complex numbers), beyond which no conventional algebraic solutions exist, as Abel and Galois established, there is an Ultima Thule of numerical categories, the **transcendental numbers**, that aren't solutions of any conceivable polynomial equation whatsoever.

05.21] An **algebraic number of degree n** is defined as being a solution of some n'th-degree polynomial equation $ax^n + bx^{n-1} + cx^{n-2} + \ldots + k = 0$

05.22] A **non-algebraic** real number is described as **transcendental**. The vast majority of conceivable real numbers are said to be in this category, including π (Loschmidt number) and e (Euler number).

06.00] In summary, the **real numbers** comprise

- Mixed numbers that are improper **rationals** with fractional parts that either terminate or recur. **Integers**, lacking fractional parts, can justifiably be included in this category

- Mixed numbers that are hybrids of integers and fractional parts that neither terminate nor recur. They may be (irrational) **algebraic** or **transcendental**, but it's hard to tell the difference even in daylight. 99.9% of us will never even need to know the difference.

(The numbers **pi**, **gamma** and **e** are transcendental, but **phi** is merely irrational, as it's the solution to a quadratic (ie 2nd-degree) polynomial equation.)

They are all categorised as **real numbers**, an unholy trinity, and jostle for position on the so-called **real number line**. The imaginary numbers likewise jostle for position on the **imaginary number line**, as they are after all **real number** multiples of i = √(-1).

It's a bit of a dogs' breakfast, though computationally of course, whether with pen and paper or in electronic calculator or computer format, we invariably deal with integers or mixed numbers whose fractional parts are truncated (decimal) representations that are ipso facto terminating.

And unless an irrational can be identified with a recognisable computational procedure, with a label such as √2 or π (pi) or φ (the Golden Ratio) or γ (the Euler-Mascheroni constant) or e (the Euler number), it is entirely anonymous.

Imagine an irrational cocktail party, at which the invitees attempted to introduce themselves to one another – nobody would ever get off first base, as their fractional id's would take forever to enounce.

And likewise, an irrational fraction can't even be catalogued, as there is nowhere to start and no itinerary to follow (as of course there are with rationals, as Cantor pointed out) – no possible Λ (listing) calculus.

07.00] Combinations a + ib of **real numbers** a with **imaginary numbers** ib, where i = √(-1) are known as **complex numbers**.

07.10] They were allegedly first encountered in the solution of cubic and quartic equations by the mediaeval Italian algebraists, but I simply don't believe that – they can equally well occur in the solution of quadratic equations, as Islamic algebraists such as Omar Khayyam would surely have known (though he would wisely have chosen to keep quiet about such theocratically heretical notions).

08.00] Up until the 19^{th} century, complex numbers could equally well be represented algebraically as

$$a + ib \text{ or } (a, b) \text{ or } re^{i\theta}, \text{ where } i^2 = -1,$$

depending on the particular context.

08.10] But then a burst of creative generalisation began, which continues to this day. In particular, in 1843, the polymathic genius and serial alcoholic William Rowan Hamilton devised **quaternions** (as alternatives to the later and more popular notion of **vectors**), which can be represented algebraically as

$$z = a + ib_1 + jb_2 + kb_3 \text{ or } (a, b_1, b_2, b_3)$$

where

$$i^2 = j^2 = k^2 = ijk = -1$$

$$ij = k \quad ji = -k$$
$$jk = i \quad kj = -i$$
$$ki = j \quad ik = -j$$

Such anticommutative multiplications were probably unprecedented at that time.

08.20] And, in that same year 1843, Hamilton's associate John Thomas Graves devised the concept of **octonions,** which can be represented algebraically as

$$z = a + b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 + b_6e_6 + b_7e_7$$

or $(a, b_1, b_2, b_3, b_4, b_5, b_6, b_7)$,

and have recently emerged as mathematical constructs ideally suited for the latest Theories of Everything. Please consult Wikipedia for the details of their multiplication tables !

08.30] There is even talk of a need for **sedenions** (yes, sixteen components). But at every step, there is of course a loss of visualisability, and indeed credibility.

08.40] More prosaically, it's high time for the terminology to be cleaned up.

- Should the original 2D complex numbers be re-branded as **duonians**, so that **all** these genera (2D, 4D, 8D, 16D) can be categorised as **complex numbers** ?

- Or should the 2D genus be left as it is, and the higher (4D, 8D, 16D) genera be categorised as **hyper-complex** numbers ?

09.00] Leibnitz (sic), whose most genuine claim to permanent fame is the fundamental formula for evaluating determinants, also came up with the Principle of Sufficient Reason (nowadays at complete odds with the ideas of quantum mechanics, which asserts that things can happen for no reason whatsoever) by which he meant that there had to be a pre-eminent reason for a thing to happen one way rather than any other way.

09.10] It's an immensely persuasive idea, but also very controversial, and I'm always bedevilled by it when hanging a picture on a wall – if there's a range of possible positions for it, which do I choose, and on what criteria ?

09.20] Leibnitz asserted that God, for whose existence he'd already found a sufficient reason, wouldn't thereafter act in any particular way without a sufficient reason either. And this of course implies that the universe is optimal, and that everything is for the best in this best of all possible worlds. Voltaire did indeed intend his character Dr Pangloss to satirise Leibnitz.

09.30] More to the point of this essay, is how did Creator Mundi choose the values of the cosmic physical constants when preparing for the Big Bang ? Was there indeed a principle of sufficient reason for each and every one ? And which of them had highest priority for the others to have to accommodate themselves to ? Were there multiple failed tests beforehand (think Salvarsan 606, or Dyson G-force 5127) ?

Surely integers and rational fractions, clearly identifiable, are far more susceptible to the principle of sufficient reason than those slippery customers the anonymous irrationals and transcendentals ? And indeed, the two highest-profile transcendentals, π (the Loschmidt number) and e (the Euler number) can be exactly formulated from integers

09.40] I remember at the age of about 14 being deeply impressed by Bohr's ludicrously naïve model of the hydrogen atom, which nevertheless revealed that the previously empirical Rydberg constant for the spectrum of atomic hydrogen was actually a combination of well-known physical constants

$$R = \frac{me^4}{8\varepsilon^2 h^3 c}$$

Never mind that these well-known physical constants were for the most part empirical themselves. Surely the next stage of physics would be to derive them from the integers and the well-known mathematical constants ultimately constructed from integers (such as pi and e) or the mysterious dimensionless fine-structure constant 1/137 (known as **alpha**) ?

09.50]   Sadly alpha isn't quite rational, and the efforts of Arthur Eddington, an astronomer and astrophysicist of immense prestige (asked in 1919 whether it was true that only three people in the world understood Einstein's theory of general relativity, he allegedly enquired, "And who's the third?"), a devout and immensely cerebral Quaker, to compute the values of the fundamental physical constants from first principles fizzled out following his early death.

Efforts such as his, and those of Einstein in his later years, to reconcile the forces of nature (that they knew about at those times), foundered on their unawareness of ongoingly deeper observational probes into the nature of the macrocosm of the universe and ever-more costly experimental probes into the microcosm of the ultimate particles from which it currently seems to be built.

09.60]   As Sherlock Holmes remarked to Dr Watson at the beginning of *A Scandal in Bohemia*, "It is a capital mistake to theorise before one has data. Insensibly one begins to twist facts to suit theories, instead of theories to suit facts."

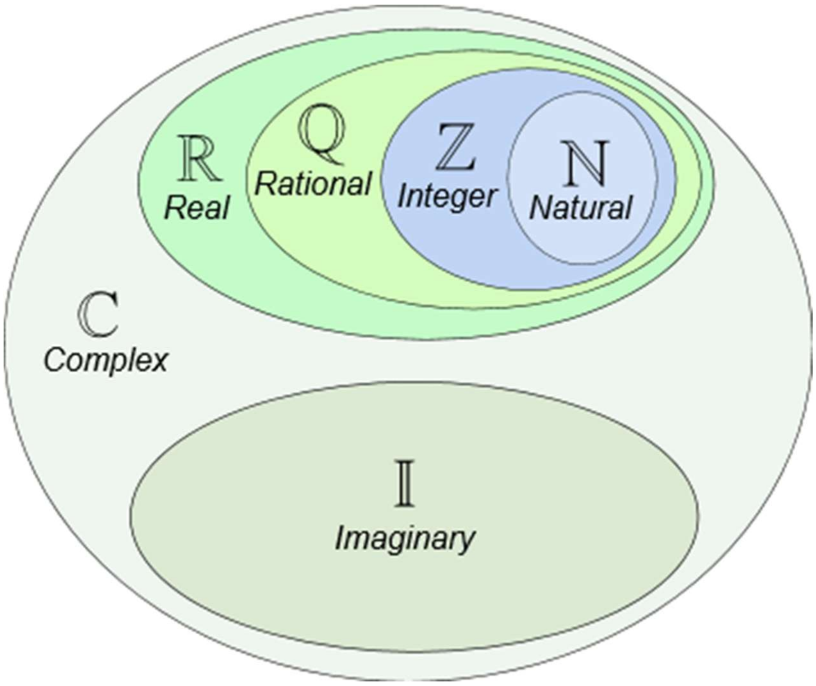
09.70]   There may be many more convolutions or indeed revolutions in the nascent theories of everything, as more data accumulate about the constant confutations of the cosmos around us, but I do believe that it will ultimately be recognised as a rational enterprise, based entirely on the something-for-nothing integers with which this rather over-lengthy essay began.

10.00] But I'm entranced by the presentation reproduced (in edited form) below. What a superb website – I only wish it had been in existence during my schooldays.

Oddly, there is no widely accepted symbol for the Irrationals, comprising the non-rational Algebraics and the Transcendentals, ie $\mathbb{R} - \mathbb{Q}$, though $\mathbb{P}$ is sometimes encountered, I believe.

https://www.mathsisfun.com/sets/number-types.html

There are sets of numbers that are used so often they have special names and symbols.



Natural (Counting, Whole) numbers are a subset of Integers Integers are a subset of Rational numbers Rational numbers are a subset of the Real numbers Combinations of Real and Imaginary numbers make up the Complex numbers.

Symbol	**Description**
$\mathbb{N}$	**Natural Numbers** The Whole numbers from 1 upwards. Or from 0 upwards in some fields of mathematics. The set is {1,2,3,...} or {0,1,2,3,...}
$\mathbb{Z}$	**Integers** The Whole numbers, {1,2,3,...}, negative Whole numbers {..., -3,-2,-1} and zero {0}. So the composite set is {..., -3, -2, -1, 0, 1, 2, 3, ...} (Z is from the German "Zahlen" meaning numbers, because I is used for the set of imaginary numbers).
$\mathbb{Q}$	**Rational Numbers** The numbers (quotients) you can make by dividing one integer by another (but not dividing by zero). In other words fractions (of a sort).
	Irrational Numbers Any Real number that is not a Rational number.
$\mathbb{A}$	**Algebraic Numbers** Any number that is a solution to a polynomial equation with rational coefficients. Includes all Rational numbers, and some Irrational numbers.
	Transcendental Numbers Any number that is not an Algebraic number Examples of transcendental numbers include π and e.
$\mathbb{R}$	**Real Numbers** All Rational and Irrational numbers. They can also be positive, negative or zero. Includes the Algebraic numbers and Transcendental numbers. They are called "Real" numbers because they are not Imaginary numbers.

$\mathbb{I}$ **Imaginary Numbers**

Numbers that when squared give a negative result.

$\mathbb{C}$ **Complex Numbers**

A combination of a Real and an Imaginary number in the form a + bi, where a and b are real, and i is imaginary.

The values a and b can be zero, so the set of Real numbers and the set of Imaginary numbers are subsets of the set of Complex numbers.

There are of course several more (in both senses) controversial numerical categories not covered by this tabulation – the Hyper-complex numbers, the Infinities and the Infinitesimals.

But, personally, I don’t for a minute believe in infinities and infinitesimals as achievable entities anyway. And I have Gauss on my side.